Some Ext-groups in categories of functors

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Our purpose is to give methods of computation of some Ext-groups in categories of functors. Let us first define the categories we will have to deal with, and explain briefly why making homological computations is so interesting in these categories.

1 The Category $\mathcal{F}$

In the whole talk, we shall work on the ground field $\mathbb{F}_2$. Most of the results can be generalized in any characteristic, except the main result (Theorem 0)

Definition 1 Let $\mathcal{E}$ be the category of $\mathbb{F}_2$-vector spaces and $\mathcal{E}^f$ its full subcategory of finite dimensional $\mathbb{F}_2$-vector spaces. The category $\mathcal{F}$ is the category of functors from $\mathcal{E}^f$ to $\mathcal{E}$. If $F \in \mathcal{F}$ takes values in $\mathcal{E}^f$, we say that F is *finite.*

The n-th symmetric power S^n, the n-th exterior power Λ^n and the n-th divided power Γ^n are examples of objects of $\mathcal{F}$. As a special case, one gets, for $n = 1$, the inclusion functor from $\mathcal{E}^f$ to $\mathcal{E}$. Let us denote it Id.

If F and G are objects of $\mathcal{F}$, one can define their tensor product $F \otimes G$ by $(F \otimes G)(V) = F(V) \otimes G(V)$. Moreover, if F is finite, one can consider the composition $G \circ F$.

In the category $\mathcal{F}$, we have a notion of *degree*, as defined by Eilenberg and MacLane (see [4]). A polynomial functor is then a functor with finite degree, and an analytic functor is a functor which is equal to the colimit of its polynomial subfunctors. This gives a full subcategory $\mathcal{F}_\omega$ of $\mathcal{F}$, consisting of analytic functors. For example, T^n, S^n and Λ^n lie in $\mathcal{F}_\omega$. This category has enough injective, so one can do homological computations in it. In order to simplify notation, we will write $\mathrm{Ext}_{\mathcal{F}}$ instead of $\mathrm{Ext}_{\mathcal{F}_\omega}$ for extensions in the category $\mathcal{F}_\omega$.

The reason why we are so interested in homological computations in $\mathcal{F}$ is that this category is strongly related to the category $\mathcal{U}$ of unstable modules over the Steenrod algebra. Indeed, Henn, Lannes and Schwartz proved that $\mathcal{F}_\omega$ is equivalent to the quotient category $\mathcal{U}/\mathcal{N}il$, where $\mathcal{N}il$ denotes the category of nilpotent modules (see [3]).

Through this equivalence, the Ext-groups $\mathrm{Ext}^*_{\mathcal{F}}(\Gamma^2, S^n \circ S^m)$ appear in the study of injective resolutions of of the cohomology of Eilenberg-MacLane spaces. These groups are still out of reach. A first step in this direction has been achieved with the following theorem, which is the main theorem of this talk.

Theorem 0 (see [5]) *If n or m is not a power of 2, then $\mathrm{Ext}^*_{\mathcal{F}}(Id, S^n \circ S^m)$ is zero. Otherwise, let $n = 2^k$ and $m = 2^h$. The Poincaré series $\varphi_{h,k}(t)$ of $\mathrm{Ext}^*_{\mathcal{F}}(Id, S^{2^h} \circ S^{2^k})$ is given by :*

$$\varphi_{h,k}(t) = \frac{1}{1 - t^{2^{h+k+1}}} \cdot \frac{\prod_{i=1}^{h+k}(1 - t^{2^i - 1})}{\prod_{i=1}^{h}(1 - t^{2^i - 1}) \cdot \prod_{i=1}^{k}(1 - t^{2^i - 1})}.$$

2 The Category of Strict Polynomial Functors

We now introduce another category of functors, which has been defined by Friedlander and Suslin, in order to relate computations in $\mathcal{F}$ to rational cohomology of group schemes (see [2]). They had to adapt some known computations from $\mathcal{F}$ to $\mathcal{P}$. During this process, they noticed that in fact, calculations were much easier in $\mathcal{P}$ than in $\mathcal{F}$. That is the reason why we introduce here this category $\mathcal{P}$: this is the right category to make our computations. We will have then to use a comparison theorem between Ext-groups in $\mathcal{P}$ and Ext-groups in $\mathcal{F}$ to get the final result.

For two vector spaces V and W, define a *strict* polynomial morphisms from V to W to be an element of $S^*(V^*) \otimes W$. The set of strict polynomial functors from V to W is denote by $\mathrm{Hom}_{pol}(V,W)$. One has a forgetful map

$$\mathrm{Hom}_{pol}(V,W) \longrightarrow \mathrm{Map}(V,W) \tag{1}$$

Definition 2 A *strict polynomial functor* P is :
- a map $V \mapsto P(V)$ from objects of $\mathcal{E}^f$ to objects of $\mathcal{E}^f$;
- for each couple (V,W) of elements of $\mathcal{E}^f$, an strict polynomial morphism $P_{V,W}$ from $\mathrm{Hom}(V,W)$ to $\mathrm{Hom}(P(V),P(W))$, *i.e.* $P_{V,W} \in \mathrm{Hom}_{pol}(\mathrm{Hom}(V,W),\mathrm{Hom}(P(V),P(W)))$;

such that
- $P_{V,V}(id_V) = id_{P(V)}$;
- $(P_{V,W})$ are compatible with composition (in the usual sense).

We say that P is homogeneous of degree d if each $P_{V,W}$ lie in $S^d(\mathrm{Hom}(V,W)^*)\otimes\mathrm{Hom}(P(V),P(W))$.

Let $\mathcal{P}$ be the category of strict polynomial functors (morphisms being natural transformations in the usual sense), and $\mathcal{P}_d$ its full subcategory of homogeneous functors of degree d. The forgetful map (1) allows to define a forgetful functor U from $\mathcal{P}$ to $\mathcal{F}$

Example 1 Aside from the symmetric, exterior and divided powers, an important example of strict polynomial functor is the *Frobenius twist* Tw. It is a functor in $\mathcal{P}_2$, defined on objects by $\mathrm{Tw} : V \longmapsto V$, and on maps by mapping $f \in \mathrm{Hom}(V,W)$ to its formal square. As a usual functor, Tw is just the identity functor.

Definition 3 Let F be an object of $\mathcal{P}$.
- Define $F^{(1)}$ the first Frobenius twist of F to be $F \circ \mathrm{Tw}$.
- Define inductively $F^{(n)}$ to be $F^{(n-1)} \circ \mathrm{Tw}$.

Remark 1
- If F is homogeneous of degree d, then $F^{(n)}$ is homogeneous of degree $2^n d$.
- As usual functors, all $F^{(n)}$ are equal to F.

3 A very short glimpse of the proof of Theorem 0

The starting point of the proof is the following computation of Franjou, Lannes and Schwartz.

Theorem 1 (Franjou, Lannes, Schwartz, see [1]) *As a Yoneda algebra,* $\mathrm{Ext}^*_{\mathcal{F}}(Id,Id)$ *is generated by elements* $e_\ell \in \mathrm{Ext}^{2^\ell}_{\mathcal{F}}(Id,Id)$, $\ell > 0$, *with commutativity relations, and* $e_\ell^2 = 0$. *As a Yoneda module* $\mathrm{Ext}^*_{\mathcal{F}}(Id,S^{2^h})$ *is the quotient of* $\mathrm{Ext}^*_{\mathcal{F}}(Id,Id)$ *by the ideal generated by* $e_1,\dots,e_h$.

Friedlander and Suslin proved that the same holds for the algebras $\mathrm{Ext}^*_{\mathcal{P}}(Id^{(r)},Id^{(r)})$ and the modules $\mathrm{Ext}^*_{\mathcal{P}}(Id^{(r)},S^{2^h(r-h)})$, except that in this case, there is only a finite number of generators e_ℓ, $1 \leqslant \ell \leqslant h$.

Our computation closely follows the proof of Franjou, Lannes and Schwartz, essentially using the same arguments : we consider hypercohomology spectral sequences of well-chosen complexes, and simplify them thanks to an important vanishing result of Pirashvili.

Theorem 2 (Pirashvili) *Let F and G be objects either of $\mathcal{F}$ or of $\mathcal{P}$. Let A be an additive functor (for example $A = Id$ in $\mathcal{F}$, or $A = Id^{(r)}$ in $\mathcal{P}$). Suppose moreover that $F(0) = 0 = G(0)$. Then* $\mathrm{Ext}^*(A, F \otimes G) = 0$.

However, we need some more ingredients to be able to achieve the computations. These ingredients we now state in the three following theorems, proved in [6].

Theorem 3 *Let $F \in \mathcal{F}$ be the image by the forgetful functor of a functor in $\mathcal{P}_n$, also denoted by F.*

- *If n is not a power of 2, then* $\mathrm{Ext}^*_{\mathcal{F}}(Id, F) = 0$.
- *If $n = 2^h$, then we have equalities of Yoneda modules :*

$$\mathrm{Ext}^*_{\mathcal{P}}(Id^{(h+r)}, F^{(r)}) = \mathrm{Ext}^*_{\mathcal{P}}(Id^{(h)}, F) \otimes \mathrm{Ext}_{\mathcal{P}}(Id^{(h+r)}, S^{2^h(r)}),$$

$$\mathrm{Ext}^*_{\mathcal{F}}(Id, F) = \mathrm{Ext}^*_{\mathcal{P}}(Id^{(h)}, F) \otimes \mathrm{Ext}_{\mathcal{F}}(Id, S^{2^h}).$$

Theorem 4 *Let P be a polynomial functor without constant term (i.e. its value on* 0 *is* 0*), and $\mathcal{C}^\bullet$ a complex of finite objects of $\mathcal{F}$, all without constant term. Then, for all $n \geqslant 0$:*

$$\mathrm{Ext}^*_{\mathcal{F}}(Id, H_n(P(\mathcal{C}^\bullet))) = \mathrm{Ext}^*_{\mathcal{F}}(Id, P(H_n(\mathcal{C}^\bullet))).$$

The same holds in the category $\mathcal{P}$.

Theorem 5 *Let F and G be two homogeneous objects of $\mathcal{P}$, respectively of degree 2^h and 2^k. Let us also denote by F and G their image in $\mathcal{F}$. Let us assume that the module structures of* $\mathrm{Ext}^*_{\mathcal{P}}(Id^{(h)}, F)$ *and* $\mathrm{Ext}^*_{\mathcal{P}}(Id^{(k)}, G)$ *are* trivial. *Then we have an isomorphism of Yoneda modules*

$$\mathrm{Ext}^*_{\mathcal{P}}(Id^{(h+k)}, G \circ F) = \mathrm{Ext}^*_{\mathcal{P}}(Id^{(h)}, F) \otimes \mathrm{Ext}^*_{\mathcal{P}}(Id^{(k)}, G) \otimes \mathrm{Ext}^*_{\mathcal{P}}(Id^{(h+k)}, S^{2^k} \circ S^{2^h}).$$

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