Some Ext-groups in categories of functors

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The category $\mathcal{F}$ – Examples

Definition 1

- $\mathcal{E}$: category of $\mathbb{F}_2$-vector spaces
- $\mathcal{E}^f$: category of finite dimensional $\mathbb{F}_2$-vector spaces
- $\mathcal{F}$: category of functors $\mathcal{E}^f \to \mathcal{E}$
- $\mathcal{F}^f$: full subcategory of *finite* functors, *i.e.* taking values in $\mathcal{E}^f$.

Example 1

- $T^n : V \mapsto V^{\otimes n}$, n-th tensor power.
- $S^n : V \mapsto T^n(V)/\mathfrak{S}_n$, n-th symmetric power.
- $\Lambda^n : V \mapsto S^n(V)/(x^2 = 0)$ (in char. 2 ! ! !), n-th exterior power.
- $Id : V \mapsto V$, the inclusion ; $Id = T^1 = S^1 = \Lambda^1$.
- $I(= I_{\mathbb{F}_2}) : V \mapsto \mathbb{F}_2{}^{V^*}$, injective object. More generally :
- $I_W : V \mapsto \mathbb{F}_2{}^{\mathrm{Hom}(V,W)}$, set of injective cogenerators of $\mathcal{F}$.
- For $F, G \in \mathcal{F}$, define $F \otimes G \in \mathcal{F}$ by $(F \otimes G)(V) = F(V) \otimes G(V)$.
- If F, G are in $\mathcal{F}$, $F \in \mathcal{F}^f$, you can define $G \circ F$.

Link between $\mathcal{F}$ and unstable modules

Definition 2

Define $\Delta : \mathcal{F} \to \mathcal{F}$ by $\Delta F : V \longmapsto \mathrm{Ker}(F(V \oplus \mathbb{F}_2) \to F(V))$.

Define the *degree* of $F \neq 0$ by saying that $\deg F \leqslant n$ if and only if $\Delta^{n+1} F = 0$. One says that

- F is *polynomial* if $\deg F$ is finite (or $F = 0$),
- F is *analytic* if F is the colimit of its polynomial subfunctors.

Denote by

- $\mathcal{U}$ the category of unstable modules over the Steenrod algebra $\mathcal{A}_2$
- $\mathcal{N}il$ the full subcategory of $\mathcal{U}$ of nilpotent modules (*i.e.* such that for all x, there exists an integer N such that $\mathrm{Sq}_0^N x = 0$)
- $\mathcal{F}_\omega$ the full subcategory of $\mathcal{F}$ of analytic functors.

Henn, Lannes and Schwartz proved the existence of an equivalence

$$\mathcal{U}/\mathcal{N}il \longrightarrow \mathcal{F}_\omega.$$

of *strict* polynomial morphisms from V to W, defined to be the set $S^*(V^*) \otimes W$. An object of $\mathrm{Hom}_{pol}(V, W)$ defines a usual polynomial map from V to W.

Definition 3

A *strict polynomial functor* P is the data of :

- a map $V \mapsto P(V)$ from objects of $\mathcal{E}^f$ to objects of $\mathcal{E}^f$;
- for each couple (V, W) of elements of $\mathcal{E}^f$, an strict polynomial morphism $P_{V,W}$ in $\mathrm{Hom}_{pol}(\mathrm{Hom}(V, W), \mathrm{Hom}(P(V), P(W)))$

such that

- $P_{V,V}(id_V) = id_{P(V)}$;
- $(P_{V,W})$ are compatible with composition (in the usual sense).

Let $\mathcal{P}$ be the category of strict polynomial functors (morphisms being natural transformations in the usual sense).

Injectives in $\mathcal{P}$ – S-resolutions

Proposition 1 (Friedlander, Suslin)

The strict polynomial functors $S^{i_1} \otimes \cdots \otimes S^{i_k}$ give a set of injective cogenerators of $\mathcal{P}$. If one restricts to such functors satisfying $i_1 + \cdots + i_k = d$, one get a set of injective cogenerators of $\mathcal{P}_d$, the full subcategory of homogeneous *functors of degree d.*

Definition 4

Let F in $\mathcal{P}_d$. Let also denote F its image in $\mathcal{F}$.

- An *S-resolution* of F in $\mathcal{P}$ is an injective resolution constructed with the set of cogenerators described above : each term of the resolution is a direct sum of tensor products of symmetric powers, of total degree d.
- An *S-resolution* of F in $\mathcal{F}$ is the image by the forgetful functor of an S-resolution in $\mathcal{P}$. It is NOT an INJECTIVE resolution.

The Frobenius twist

Definition 5

The *Frobenius twist* Tw is an object of $\mathcal{P}_2$, defined on objects by $\mathrm{Tw} : V \longmapsto V$, and on maps by

$$(\mathrm{Tw})_{V,W} = \sum (f_i^*)^2 \otimes f_i \in S^2(\mathrm{Hom}(V, W)^*) \otimes \mathrm{Hom}(V, W).$$

Definition 6

Let F be an object of $\mathcal{P}$.

- Define $F^{(1)}$ the first Frobenius twist of F to be $F \circ \mathrm{Tw}$.
- Define inductively $F^{(n)}$ to be $F^{(n-1)} \circ \mathrm{Tw}$.

Remark 1

- If F is homogeneous of degree d, then $F^{(n)}$ is homogeneous of degree $2^n d$.
- As a usual functor, Tw is equal to the identity functor Id.
- Therefore, as usual functors, all $F^{(n)}$ are equal to F.

The main theorem

The main theorem consists in the computation of the groups $\mathrm{Ext}^*_{\mathcal{F}}(Id, S^n \circ S^m)$.

Theorem 0

If n or m is not a power of 2, then $\mathrm{Ext}^*_{\mathcal{F}}(Id, S^n \circ S^m)$ is zero. Otherwise, let $n = 2^k$ and $m = 2^h$. The Poincaré series $\varphi_{h,k}(t)$ of $\mathrm{Ext}^*_{\mathcal{F}}(Id, S^{2^h} \circ S^{2^k})$ is given by :

$$\varphi_{h,k}(t) = \frac{1}{1 - t^{2^{h+k+1}}} \cdot \frac{\prod_{i=1}^{h+k} (1 - t^{2^i - 1})}{\prod_{i=1}^{h} (1 - t^{2^i - 1}) \cdot \prod_{i=1}^{k} (1 - t^{2^i - 1})}.$$

As we will see later, this implies the knowledge of similar Ext-groups in the category $\mathcal{P}$.

and Schwartz to compute $\mathrm{Ext}(Id, S^n)$, using essentially the same tools. However, we need some more tools.

Theorem 1

A comparison theorem between Ext*-groups in $\mathcal{P}$ and* Ext*-groups in $\mathcal{F}$ when the left variable is Id.* Ext*-groups in one category completely determine* Ext*-groups in the other.*

Theorem 2

However not exact, the post-composition is almost exact, when dealing with $\mathrm{Ext}(Id, -)$*-groups.*

Theorem 3

A formula giving (for very special F and G) $\mathrm{Ext}^*(Id, G \circ F)$ *knowing* $\mathrm{Ext}^*(Id, F)$ *and* $\mathrm{Ext}^*(Id, G)$.

Calculations of $\mathrm{Ext}(Id, S^m)$

Theorem 4 (Franjou, Lannes, Schwartz)
If m is not a power of 2, $\mathrm{Ext}^*_{\mathcal{F}}(Id, S^m) = 0$. If $m = 2^h$, then :

$$\mathrm{Ext}^k_{\mathcal{F}}(Id, S^{2^h}) = \begin{cases} \mathbb{F}_2 & \text{if } k \equiv 0 \mod 2^{h+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5 (Friedlander, Suslin)
For any $r \geqslant h$,

$$\mathrm{Ext}^k_{\mathcal{P}}(Id^{(r)}, S^{2^h(r-h)}) = \begin{cases} \mathbb{F}_2 & \text{if } k \equiv 0 \mod 2^{h+1},\ k < 2^{r+1} \\ 0 & \text{otherwise.} \end{cases}$$

Yoneda structures on $\mathrm{Ext}(Id, S^n)$

As a Yoneda algebra, $\mathrm{Ext}^*_{\mathcal{F}}(Id, Id)$ is generated by elements $e_\ell \in \mathrm{Ext}^{2^\ell}_{\mathcal{F}}(Id, Id)$, $\ell > 0$, with commutativity relations, and $e_\ell^2 = 0$. As a Yoneda module $\mathrm{Ext}^*_{\mathcal{F}}(Id, S^{2^h})$ is the quotient of $\mathrm{Ext}^*_{\mathcal{F}}(Id, Id)$ by the ideal generated by $e_1, \ldots, e_h$.

The same holds for the Yoneda algebra $\mathrm{Ext}^*_{\mathcal{P}}(Id^{(r)}, Id^{(r)})$ and the Yoneda modules $\mathrm{Ext}^*_{\mathcal{P}}(Id^{(r)}, S^{2^h(r-h)})$, except that in this case, there are only generators e_ℓ for $\ell \leqslant r$.

Remark 2
A result of Friedlander and Suslin (generalized by Totaro in any characteristic and for more general functors) says that the projective dimension of $Id^{(r)}$ is $2^{r+1} - 2$. Hence, $\mathrm{Ext}^*_{\mathcal{P}}(Id^{(r)}, F)$ is always zero when $* > 2^{r+1} - 2$. Therefore one can see $\mathrm{Ext}^*_{\mathcal{P}}(Id^{(r)}, F)$ as a module over $\mathrm{Ext}^*_{\mathcal{P}}(Id^{(r')}, Id^{(r')})$, $r' \geqslant r$, or over $\mathrm{Ext}^*_{\mathcal{F}}(Id, Id)$, by letting the action of e_ℓ, $\ell > r$ be trivial.

A comparison theorem between $\mathcal{F}$ and $\mathcal{P}$

Theorem 1
Let $F \in \mathcal{F}$ be the image by the forgetful functor of a functor, also denoted by F, in $\mathcal{P}_n$.
- *If n is not a power of 2, then $\mathrm{Ext}^*_{\mathcal{F}}(Id, F) = 0$.*
- *If $n = 2^h$, then we have equalities of Yoneda modules :*

$$\mathrm{Ext}^*_{\mathcal{P}}(Id^{(h+r)}, F^{(r)}) = \mathrm{Ext}^*_{\mathcal{P}}(Id^{(h)}, F) \otimes \mathrm{Ext}_{\mathcal{P}}(Id^{(h+r)}, S^{2^h(r)}),$$

$$\mathrm{Ext}^*_{\mathcal{F}}(Id, F) = \mathrm{Ext}^*_{\mathcal{P}}(Id^{(h)}, F) \otimes \mathrm{Ext}_{\mathcal{F}}(Id, S^{2^h}).$$

Let $\mathcal{C}^\bullet : \mathcal{C}^0 \to \mathcal{C}^1 \to \cdots$ be a complex in $\mathcal{F}$ (or $\mathcal{P}$, or any abelian category with enough injectives). Following Cartan and Eilenberg, we can construct an injective resolution $I^{\bullet\bullet}$ of the complex $\mathcal{C}^\bullet$, with good properties. Applying the functor $\mathrm{Hom}(T, -)$, we get a bicomplex $\mathrm{Hom}(T, I^{\bullet\bullet})$, which we can filter either horizontally or vertically. This give two spectral sequences $\mathbf{I}(T, \mathcal{C}^\bullet)$ (first hypercohomology spectral sequence) and $\mathbf{II}(T, \mathcal{C}^\bullet)$ (second hypercohomology spectral sequence) with the following properties :
- the two hypercohomogy spectral sequences have same abutment, since they are defined from filtrations of the same object. They of course do not define the same filtration on the abutment ;
- $\mathbf{I}_1^{s,*}(T, \mathcal{C}^\bullet) = \mathrm{Ext}^*(T, \mathcal{C}^s)$;
- $\mathbf{II}_2^{*,t}(T, \mathcal{C}^\bullet) = \mathrm{Ext}^*(T, H^t(\mathcal{C}^\bullet))$;
- differentials of rank r are of bidegree $(r, 1-r)$.

Pirashvili's vanishing theorem

Theorem 6 (Pirashvili)
Let F and G be objects either of $\mathcal{F}$ or of $\mathcal{P}$. Let A be an additive functor (for example $A = Id$ in $\mathcal{F}$, or $A = Id^{(r)}$ in $\mathcal{P}$). Suppose moreover that $F(0) = 0 = G(0)$. Then

$$\mathrm{Ext}^*(A, F \otimes G) = 0.$$

There exist generalizations of this fact when A is not additive any more. See Franjou, Friedlander, Suslin, Scorichenko.

The complex $\mathcal{S}^\bullet$

Let $\mathcal{S}^\bullet(V)$ be the reduced cobar-construction of the coaugmented Hopf algebra $S^*(V)$. It defines a complex in the category $\mathcal{P}$:

$$\mathcal{S}^\bullet \ :\ 0 \longrightarrow \overline{S^*} \longrightarrow \overline{S^*}^{\otimes 2} \longrightarrow \cdots \longrightarrow \overline{S^*}^{\otimes k} \longrightarrow \cdots,$$

where $\overline{S^*} = S^{*\geqslant 1}$.

A classical computation gives

$$\mathrm{H}^*(\mathcal{S}^\bullet) = \bigotimes_{h \geqslant 0} S^{*(h)}.$$

The complexes $\mathcal{S}^\bullet_n$

Taking the homogeneous part of degree n of the complex $\mathcal{S}^\bullet$, we get a complex $\mathcal{S}^\bullet_n$:

$$S^n \longrightarrow \bigoplus_{\substack{i_1+i_2=n \\ i_1,i_2>0}} S^{i_1} \otimes S^{i_2} \longrightarrow \bigoplus_{\substack{i_1+i_2+i_3=n \\ i_1,i_2,i_3>0}} S^{i_1} \otimes S^{i_2} \otimes S^{i_3} \longrightarrow \cdots \longrightarrow (S^1)^{\otimes n},$$

whose cohomology is the homogeneous part of degree n of the cohomology of $\mathcal{S}^\bullet$. Hence,

$$\mathrm{H}^t(\mathcal{S}^\bullet_n) = \bigoplus_{\substack{\sum_{\ell \geqslant 0} i_\ell = t \\ \sum_{\ell \geqslant 0} i_\ell 2^\ell = n}} \bigotimes_{\ell \geqslant 0} S^{i_\ell(\ell)}$$

Proposition 2
The square $S^{2^h} \to S^{2^{h+1}}$ induces a morphism of complexes $\mathcal{S}^\bullet_{2^h} \to \mathcal{S}^\bullet_{2^{h+1}}$, which induces inclusion on E_2-terms of the spectral sequences. For all $k \geqslant h$ and $r \leqslant 2^h + 1$, we get an equality

$$\mathbf{II}_r^{*,*\leqslant 2^h}(Id, G \circ \mathcal{S}^\bullet_{2^k} \circ F) = \mathbf{II}_r^{*,*\leqslant 2^h}(Id, G \circ \mathcal{S}^\bullet_{2^h} \circ F).$$

The right handside for $r = 2^h + 1$ is also equal to the E_∞-term of the same spectral sequence.

Remark 3
The proposition also implies that the differential of rank $r \leqslant 2^h$ of the spectral sequence $\mathbf{II}(Id, G \circ \mathcal{S}^\bullet_{2^k} \circ F)$ are the same as those of $\mathbf{II}(Id, G \circ \mathcal{S}^\bullet_{2^h} \circ F)$.

A FORMULA FOR $\mathrm{Ext}^*(Id, G \circ F)$

THEOREM 3
Let F and G be two homogeneous objects of $\mathcal{P}$, respectively of degree 2^h and 2^k. Let us also denote by F and G their image in $\mathcal{F}$. Let us assume that the module structures of $\mathrm{Ext}^*_{\mathcal{P}}(Id^{(h)}, F)$ and $\mathrm{Ext}^*_{\mathcal{P}}(Id^{(k)}, G)$ are trivial. *Then we have an isomorphism of Yoneda modules*

$$\mathrm{Ext}^*_{\mathcal{P}}(Id^{(h+k)}, G \circ F) =$$
$$= \mathrm{Ext}^*_{\mathcal{P}}(Id^{(h)}, F) \otimes \mathrm{Ext}^*_{\mathcal{P}}(Id^{(k)}, G) \otimes \mathrm{Ext}^*_{\mathcal{P}}(Id^{(h+k)}, S^{2^k} \circ S^{2^h}).$$

REMARK 4
According to the comparison theorem, this also gives a description of the Yoneda module $\mathrm{Ext}^*_{\mathcal{F}}(Id, G \circ F)$.

A DESCRIPTION OF THE DIFFERENTIALS OF $\mathbf{II}(Id, \mathcal{S}^{\bullet}_{2^k} \circ S^h)$

PROPOSITION 3
The differentials of the spectral sequence $\mathbf{II}(Id, \mathcal{S}^{\bullet}_{2^k} \circ S^{2^h})$ coming from the 2^k-th (upper) line are
– *zero if their target is of first degree*

$$s \equiv 0, 1 \dots, 2^{h+k} - 1 \mod 2^{h+k+1},$$

– *surjective if their target is of first degree*

$$s \equiv 2^{h+k}, \dots, 2^{h+k+1} - 1 \mod 2^{h+k+1}.$$

LEMMA 1
The elements $e_1, \dots, e_{h+k}$ of $\mathrm{Ext}^*_{\mathcal{F}}(Id, Id)$ act trivially on $\mathrm{Ext}^*_{\mathcal{F}}(Id, S^{2^k} \circ S^{2^h})$.

THE POST-COMPOSITION IS ALMOST EXACT

THEOREM 2
Let P be a polynomial functor in $\mathcal{F}$, without constant term, i.e. $P(0) = 0$. Let further $0 \to F' \to F \to F'' \to 0$ be a short exact sequence of finite functors without constant terms. Denote by H the cohomology at $P \circ F$ of the complex :

$$0 \longrightarrow P \circ F' \longrightarrow P \circ F \longrightarrow P \circ F'' \longrightarrow 0.$$

Then $\mathrm{Ext}^*_{\mathcal{F}}(Id, H) = 0$.

COROLLARY 1
Let P be a polynomial functor without constant term, and $\mathcal{C}^{\bullet}$ a complex of finite objects of $\mathcal{F}$, all without constant term. Then, for all $n \geqslant 0$:

$$\mathrm{Ext}^*_{\mathcal{F}}(Id, H_n(P(\mathcal{C}^{\bullet}))) = \mathrm{Ext}^*_{\mathcal{F}}(Id, P(H_n(\mathcal{C}^{\bullet}))).$$