MacLane cohomology with coefficients in composed functors

Alain Troesch

LAGA, Université Paris 13

Extensions involving composed functors

Alain Troesch

LAGA, Université Paris 13

The category $\mathcal{P}$ of strict polynomial functors

p : a prime number
$\mathcal{E}^f$: category of finite $\mathbb{F}_p$-vector spaces

Definition (Friedlander, Suslin)
A *strict polynomial functor* P is :
- a map $V \mapsto P(V)$ from $\mathrm{Ob}(\mathcal{E}^f)$ to $\mathrm{Ob}(\mathcal{E}^f)$;
- for each pair (V,W) of objects of $\mathcal{E}^f$, a *strict* polynomial map from $\mathrm{Hom}(V,W)$ to $\mathrm{Hom}(P(V),P(W))$;

such that $P_{V,V}(id_V) = id_{P(V)}$ and the family $(P_{V,W})$ is compatible with composition (in the usual sense).

$\mathcal{P}$: category of strict polynomial functors.
$\mathcal{P}_d$: subcategory of *homogeneous* functors of degree d.

- $T^n : V \mapsto V^{\otimes n}$, n-th tensor power
- $S^n : V \mapsto T^n(V)/\mathfrak{S}_n$, n-th symmetric power
- $\Lambda^n : V \mapsto T^n(V)/(x\otimes x = 0, x\otimes y = -y\otimes x)$, n-th exterior power
- $\mathrm{Id} : V \mapsto V$, the identity ; $\mathrm{Id} = T^1 = S^1 = \Lambda^1$
- Tw : Frobenius twist, defined as :
  identity on objects, power p map on morphisms.
  As a usual functor, $\mathrm{Tw} = \mathrm{Id}$
- $F^{(1)} = F \circ \mathrm{Tw}$, first Frobenius twist of F
- $F^{(n)} = F^{(n-1)} \circ \mathrm{Tw}$, n-th Frobenius twist of F

$T^n, S^n, \Lambda^n \in \mathcal{P}_n$ $\quad \mathrm{Id} \in \mathcal{P}_1$ $\quad \mathrm{Tw} \in \mathcal{P}_p$ $\quad F \in \mathcal{P}_d \Rightarrow F^{(n)} \in \mathcal{P}_{p^n d}$

Injectives in $\mathcal{P}$

Proposition (Friedlander, Suslin)
The strict polynomial functors $S^{i_1} \otimes \cdots \otimes S^{i_k}$ form a set of injective cogenerators of $\mathcal{P}$.

Such functors satisfying $i_1 + \cdots + i_k = d$ form a set of injective cogenerators of $\mathcal{P}_d$.

Consequences

1. enough injectives $\Longrightarrow$ existence of Ext-groups in $\mathcal{P}$
2. Each object F of $\mathcal{P}_d$ admits an injective resolution the terms of which are direct sums of tensor products of symmetric powers, of total degree d.

Examples of Ext-groups in $\mathcal{P}$

Theorem (Pirashvili's vanishing theorem)
Let F and G be such that $F(0) = 0 = G(0)$, and A an additive functor. Then

$$\mathrm{Ext}^*_{\mathcal{P}}(A, F \otimes G) = 0.$$

Theorem (Friedlander, Suslin, after Franjou, Lannes, Schwartz)

$$\mathrm{Ext}^k_{\mathcal{P}}(\mathrm{Id}^{(h+\ell)}, S^{p^h (\ell)}) = \begin{cases} \mathbb{F}_p & \text{if } k \equiv 0 \mod 2p^h, \quad k < 2p^{h+\ell} \\ 0 & \text{otherwise.} \end{cases}$$

Aim of the talk

Question
Does there exist a formula giving the module $\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(?)}, G \circ F)$ in terms of the modules $\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(?)}, F)$ and $\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(?)}, G)$?

Remark
module = module over the algebra $\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(?)}, \mathrm{Id}^{(?)})$;
product = Yoneda composition of extensions.

We give a fomula for F and G satisfying a certain hypothesis. The general case is unknown.

Theorem
Let F and G be two homogeneous objects of $\mathcal{P}$, respectively of degree p^h and p^k. Let us assume that $\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(h)}, F)$ and $\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(k)}, G)$ have trivial module structure. Then we have an isomorphism of Yoneda modules

$$\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(h+k+\ell)}, (G \circ F)^{(\ell)})$$
$$\cong$$
$$\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(h)}, F) \otimes \mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(k)}, G) \otimes \mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(h+k+\ell)}, (S^{p^k} \circ S^{p^h})^{(\ell)}),$$

where $\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(h+k+\ell)}, \mathrm{Id}^{(h+k+\ell)})$ acts on the third factor of the tensor product.

Comments on the formula

Remark

The functors F and G are requested to satisfy the same hypothesis which we will call $(\mathcal{H})$:

Hypothesis $(\mathcal{H})$ for $F \in \mathcal{P}_{p^h}$:
The module structure of $\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(h)}, F)$ is trivial.

Remark

The modules $\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(h+k+\ell)}, (S^{p^k} \circ S^{p^h})^{(\ell)})$ are unknown unless $p = 2$. In this case it appears in fact as a consequence of the formula, as we will see later.

A sketch of the proof

Main ingredient : Post-composition is almost exact

Theorem

Let $P \in \mathcal{P}_{p^h}$, and $0 \to F' \to F \to F'' \to 0$ a short exact sequence of objects of $\mathcal{P}_{p^k}$. Denote by H the cohomology at $P \circ F$ of :

$$0 \longrightarrow P \circ F' \longrightarrow P \circ F \longrightarrow P \circ F'' \longrightarrow 0.$$

Then $\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(h+k)}, H) = 0$.

Corollary

Let $P \in \mathcal{P}_{p^h}$, and $\mathcal{C}^\bullet$ a complex of objects of $\mathcal{P}_{p^k}$. Then,

$$\forall n,\ \mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(h+k)}, \mathrm{H}^n(P(\mathcal{C}^\bullet))) = \mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(h+k)}, P(\mathrm{H}^n(\mathcal{C}^\bullet))).$$

Examples of functors satisfying the hypothesis $(\mathcal{H})$

Compositions of symmetric powers

In the following examples, $p = 2$.

Proposition

The compositions $S^2 \circ S^{2^h}$, $h \geqslant 0$, satisfy the hypothesis $(\mathcal{H})$, i.e. the module structure of $\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(h+1)}, S^2 \circ S^{2^h})$ is trivial.

Corollary

The functors $S^{2^{i_k}} \circ \cdots \circ S^{2^{i_1}}$ also satisfy the hypothesis $(\mathcal{H})$.

Corollary $\Longrightarrow$ description of $\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(i_1+\cdots+i_k)}, S^{2^{i_k}} \circ \cdots \circ S^{2^{i_1}})$ (induction on k and i_k, using the hypercohomology spectral sequences of the reduced Cobar construction of S^*)

Theorem

The Poincaré series of $\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(i_1+\cdots+i_k)}, S^{2^{i_k}} \circ \cdots \circ S^{2^{i_1}})$ is :

$$\varphi_{i_1,\dots,i_k}(t) = \frac{\prod_{i=1}^{i_1+\cdots+i_k}(1 - t^{2^i-1})}{\prod_{i=1}^{i_1}(1 - t^{2^i-1}) \cdots \prod_{i=1}^{i_k}(1 - t^{2^i-1})}.$$

It is a polynomial of degree $d < 2^{i_1+\cdots+i_k+1}$.

Examples among simple objects

Proposition

The Schur functors $W_{(2^h-1,1)} = \mathrm{Ker}\left(\Lambda^{2^h-1} \otimes \Lambda^1 \to \Lambda^{2^h}\right)$ satisfy $(\mathcal{H})$.

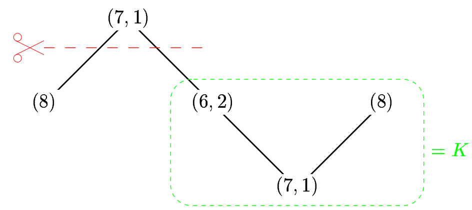
Idea : Using the short exact sequence defining $W_{(2^h-1,1)}$, one shows that the total dimension of $\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(h)}, W_{(2^h-1,1)})$ is 1, hence the module structure cannot be non-trivial.

Proposition

The simple object $S_{(3,1)}$ satisfies $(\mathcal{H})$.

Idea : $\Lambda^2 \circ \Lambda^2 \cong \Lambda^4 \oplus S_{(3,1)}$.

Idea : Representation theory of the symmetric groups gives the structure of $\Lambda^6 \otimes \Lambda^2$:



$\mathrm{Ext}^{*+1}_{\mathcal{P}}(\mathrm{Id}^{(3)}, S_{(7,1)}) = \mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(3)}, \Lambda^8) \oplus \mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(3)}, K)$, and each of the two right terms have trivial module structure because their total dimension is 1.

Further examples

Proposition

The Schur functors $W_{(6,2)}$ and $W_{(5,3)}$ *and the simple functor* $S_{(6,2)}$ *satisfy* $(\mathcal{H})$.

Question

Do all Schur and simple objects satisfy $(\mathcal{H})$ *?*
If not, which ones do ?