THE CATEGORY \( \mathcal{P} \) OF STRICT POLYNOMIAL FUNCTORS

\( p \) : a prime number

\( \mathcal{E}' \) : category of finite \( \mathbb{F}_p \)-vector spaces

**Definition** (Friedlander, Suslin)

A **strict polynomial functor** \( P \) is:

- a map \( V \mapsto \mathcal{P}(V) \) from \( \text{Ob}(\mathcal{E}') \) to \( \text{Ob}(\mathcal{E}') \);
- for each pair \( \{ V, W \} \) of objects of \( \mathcal{E}' \), a **strict** polynomial map from \( \text{Hom}(V, W) \) to \( \text{Hom}(\mathcal{P}(V), \mathcal{P}(W)) \);

such that \( P_V(id_V) = id_{\mathcal{P}(V)} \) and the family \( \{ P_V, W \} \) is compatible with composition (in the usual sense).

**Theorem**

Let \( F \) and \( G \) be two homogeneous objects of \( \mathcal{P} \), respectively of degree \( p^k \) and \( p^l \). Let us assume that \( \text{Ext}^{2k}_{\mathcal{P}}(Id^{p^k}, F) \) and \( \text{Ext}^{2l}_{\mathcal{P}}(Id^{p^l}, G) \) have trivial module structure. Then we have an isomorphism of Yoneda modules

\[ \text{Ext}^{2k}_{\mathcal{P}}(Id^{(2k+l)}(F), G) \cong \text{Ext}^{2l}_{\mathcal{P}}(Id^{(2l)}(G), F) \]

where \( \text{Ext}^{2k}_{\mathcal{P}}(Id^{(2k+l)}(F), G) \) acts on the third factor of the tensor product.

AIM OF THE TALK

**Question**

Does there exist a formula giving the module \( \text{Ext}^{2k}_{\mathcal{P}}(Id^{p^k}, G \circ F) \) in terms of the modules \( \text{Ext}^{2k}_{\mathcal{P}}(Id^{p^k}, F) \) and \( \text{Ext}^{2k}_{\mathcal{P}}(Id^{p^k}, G) \)?

**Remark**

- module over the algebra \( \text{Ext}^{2k}_{\mathcal{P}}(Id^{p^k}, Id^{p^k}) \);
- product = Yoneda composition of extensions.

We give a formula for \( F \) and \( G \) satisfying a certain hypothesis. The general case is unknown.

EXAMPLES OF EXT-GROUPS IN \( \mathcal{P} \)

**Theorem** (Pirashvili’s vanishing theorem)

Let \( F \) and \( G \) be such that \( F(0) = 0 = G(0) \), and \( A \) an additive functor. Then

\[ \text{Ext}^{2k}_{\mathcal{P}}(A, F \circ G) = 0. \]

**Theorem** (Friedlander, Suslin, after Franjou, Lannes, Schwartz)

\[ \text{Ext}^{2k}_{\mathcal{P}}(Id^{(2k+t)}, S^{p^k}(t)) = \begin{cases} \mathbb{F}_p & k \equiv 0 \mod 2p^t, \quad k < 2p^{t+1} \\ 0 & \text{otherwise}. \end{cases} \]

INJECTIVES IN \( \mathcal{P} \)

**Proposition** (Friedlander, Suslin)

The strict polynomial functors \( S^{i_1} \odot \cdots \odot S^{i_k} \) form a set of injective cogenerators of \( \mathcal{P} \).

Such functors satisfying \( i_1 + \cdots + i_k = d \) form a set of injective cogenerators of \( \mathcal{P}_d \).

CONSEQUENCES

1. enough injectives \( \Rightarrow \) existence of Ext-groups in \( \mathcal{P} \)
2. Each object \( F \) of \( \mathcal{P}_d \) admits an injective resolution the terms of which are direct sums of tensor products of symmetric powers, of total degree \( d \).
EXAMPLES OF FUNCTORS SATISFYING THE HYPOTHESIS (H)

MAIN INGREDIENT: POST-COMPOSITION IS ALMOST EXACT

THEOREM
Let $P \in \mathcal{P}_p$, and $0 \to F' \to F \to F'' \to 0$ a short exact sequence of objects of $\mathcal{P}_p$. Denote by $H$ the cohomology at $P \circ F$ of:

$$0 \to P \circ F' \to P \circ F \to P \circ F'' \to 0.$$ Then $\text{Ext}^*_p(\text{Id}^{[b+k]}, H) = 0$

COROLLARY
Let $P \in \mathcal{P}_p$, and $\mathcal{C}^*$ a complex of objects of $\mathcal{P}_p$. Then:

$$\forall n, \text{Ext}^*_p(\text{Id}^{[b+k]}, H^n(P(\mathcal{C}^*))) = \text{Ext}^*_p(\text{Id}^{[b+k]}, H^n(P(\mathcal{C}^*))).$$

A SKETCH OF THE PROOF

EXAMPLES AMONG SIMPLE OBJECTS

PROPOSITION
The Schur functors $W_{[2^i-1,1]}$ satisfy $(H)$.

Idea: Using the short exact sequence $W_{[2^i-1,1]}$ defines $W_{[2^i-1,1]}$, one shows that the total dimension of $\text{Ext}^*_p(\text{Id}^{[b]}, W_{[2^i-1,1]})$ is 1, hence the module structure cannot be non-trivial.

PROPOSITION
The simple object $S_{[3,1]}$ satisfies $(H)$.

Idea: $\Lambda^2 \circ \Lambda^2 \cong \Lambda^4 \oplus S_{[3,1]}$.

THEOREM
The Poincaré series of $\text{Ext}^*_p(\text{Id}^{[b_1+\ldots+b_k]}, S^{2^{i_1}} \circ \ldots \circ S^{2^{i_k}})$ is:

$$\varphi_{i_1,\ldots,i_k}(t) = \frac{\prod_{i=1}^{i_k} (1-t^{2^{i-1}})}{\prod_{i=1}^{i_1} (1-t^{2^{i-1}}) \cdot \ldots \cdot \prod_{i=1}^{i_k} (1-t^{2^{i-1}})}.$$ It is a polynomial of degree $d < 2^{i_1+\ldots+i_k+1}$.

COMPOSITIONS OF SYMMETRIC POWERS

In the following examples, $p = 2$.

PROPOSITION
The compositions $S^{2^i} \circ S^{2^i}$, $h \geq 0$, satisfy the hypothesis $(H)$, i.e.

The module structure of $\text{Ext}^*_p(\text{Id}^{[b+1]}, S^{2^i} \circ S^{2^i})$ is trivial.

COROLLARY
The functors $S^{2^i} \circ \ldots \circ S^{2^i}$ also satisfy the hypothesis $(H)$.

Remark: The module structure of $\text{Ext}^*_p(\text{Id}^{[i_1+\ldots+i_k]}, (S^{2^h} \circ S^{2^h})(t))$ are unknown unless $p = 2$. In this case it appears in fact as a consequence of the formula, as we will see later.

COMMENTS ON THE FORMULA

Remark
The functors $F$ and $G$ are requested to satisfy the same hypothesis which we will call $(H)$:

Hypothesis $(H)$ for $F \in \mathcal{P}_p$:

The module structure of $\text{Ext}^*_p(\text{Id}^{[b]}, F)$ is trivial.

Remark
The functors $S^{2^h} \circ \ldots \circ S^{2^i}$ also satisfy the hypothesis $(H)$.
Further Examples

Proposition
The Schur functors $W_{[6,3]}$ and $W_{[3,3]}$ and the simple functor $S_{[6,2]}$
satisfy $(\mathcal{H})$.

Question
Do all Schur and simple objects satisfy $(\mathcal{H})$?
If not, which ones do?