Decompositions of composed functors and applications to calculations of Ext-groups in functor categories

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The category $\mathcal{F}$

Definition

- p : a prime
- $\mathcal{E}$: category of $\mathbb{F}_p$-vector spaces
- $\mathcal{E}^f$: category of finite $\mathbb{F}_p$-vector spaces
- $\mathcal{F}$: category of functors $\mathcal{E}^f \longrightarrow \mathcal{E}$

Examples

- $T^n : V \mapsto V^{\otimes n}$, n-th tensor power
- $S^n : V \mapsto T^n(V)/\mathfrak{S}_n$, n-th symmetric power
- $\Gamma^n : V \mapsto T^n(V)^{\mathfrak{S}_n}$, n-th divided power
- $\Lambda^n : V \mapsto T^n(V)/(x\otimes x=0, x\otimes y=-y\otimes x)$, n-th exterior power
- $V \mapsto V$, identity functor ; $\mathrm{Id} = T^1 = S^1 = \Gamma^1 = \Lambda^1$
- $\mathbb{F}_p{}^{V^*}$, injective object in $\mathcal{F}$, $I = S^*/(x^p = x)$

Calculus in $\mathcal{F}$

Notation

$\Delta F : V \longmapsto \mathrm{Ker}\Big(F(V \oplus \mathbb{F}_p) \to F(V)\Big)$

Definition

F is *of degree at most n* iff $\Delta^{n+1}(F) = 0$.

Hence, one can define *polynomial* and *analytic* functors.

Definition

The *n-th Taylor functor* $t_n(F)$ is the greatest subobject of F of degree at most n.

Example

$t_n(I) = S^0 \oplus \cdots \oplus S^n/(x^p = x)$.

The Taylor tower of $S^n \circ I$ splits.

Applications to the calculation of $\mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, S^n \circ I)$, and (work in progress) of $\mathrm{Ext}^*_{\mathcal{F}}(\Gamma^d, S^n \circ I)$.

Calculation of $\mathrm{Ext}^*_{\mathcal{F}}(-,-)$ means :

- Determine the graded vector space, *i.e.* its *Poincaré series.*
- Determine its *Yoneda module structure* over $\mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, \mathrm{Id})$
  Product : *Yoneda composition* of extensions.

More general situation : $S^n \circ B$, where B is **boolean**.

Motivations – **1. Connections with unstable modules**

- $\mathcal{U}$: category of unstable modules over the Steenrod algebra
- $\mathcal{N}il$: full subcategory of $\mathcal{U}$ of *nilpotent objects*
- $\mathcal{F}_\omega$: full subcategory of $\mathcal{F}$ of *analytic functors*

Theorem (Henn, Lannes, Schwartz)

There exists a functor $\overline{f} : \mathcal{U} \longrightarrow \mathcal{F}$ inducing an equivalence of categories $f : \mathcal{U}/\mathcal{N}il \xrightarrow{\simeq} \mathcal{F}_\omega$. Explicitely, $\overline{f}(M)(V) = T_V(M)^0$.

Examples

- $\overline{f}(F(n)) = \Gamma^n$
- $\overline{f}(F(1)^{\otimes n}) = T^n$
- $\overline{f}(H^*W) = I_W = \mathbb{F}_p{}^{\mathrm{Hom}(-,W)}$
- $\overline{f}(H^*\mathbb{F}_p) = I$
- $\overline{f}(H^*(K(V,n))) = I_V \circ \Gamma^n$

How to rise from $\mathcal{F}$-level to $\mathcal{U}$-level

The projection $\mathcal{U} \to \mathcal{U}/\mathcal{N}il$ admits a right adjoint.
Therefore, one gets a **localization functor** $\ell : \mathcal{U} \to \mathcal{U}$.

There exists a **localization spectral sequence** :
$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{U}}(M, \ell^t(N)) \Longrightarrow \mathrm{Ext}^{s+t}_{\mathcal{F}}(\overline{f}(M), \overline{f}(N))$,
where ℓ^t stands for the t-th derived functor of ℓ.

When $M = F(n)$, **it degenerates** at rank 2, and :
$\mathrm{Hom}_{\mathcal{U}}(F(n), \ell^*(N)) = \mathrm{Ext}^*_{\mathcal{F}}(\Gamma^n, \overline{f}(N))$

Motivations – **2. Connections with GL_n-modules**

Definition

Let $\mathcal{P}$ denote the category of *strict polynomial functors.* It is defined the same way as $\mathcal{F}$ except that the behaviour of $P \in \mathcal{P}$ on maps is *strictly polynomial.* It is homogeneous of degree d if each polynomial defining P on maps is homogeneous of degree d.

Theorem (Friedlander, Suslin)

If $P, Q \in \mathcal{P}$ are homogeneous of degree d, and $n \geqslant d$,
$\mathrm{Ext}^*_{\mathcal{P}}(P,Q) \simeq \mathrm{Ext}^*_{GL_n}(P(\mathbb{F}_p{}^n), Q(\mathbb{F}_p{}^n))$.

From $\mathrm{Ext}^*(\mathrm{Id}, S^n)$ and $G \hookrightarrow GL_n$ for every finite group scheme G :

Theorem (Friedlander, Suslin)

G a finite group scheme. $\mathrm{H}^*(G, \mathbb{F}_p)$ is a finitely gen. algebra.

$F \in \mathcal{P}$, homogeneous of degree d.

- *If $d \neq p^h$, then $\mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, F) = 0$.*
- *If $d = p^h$, then $\mathrm{Ext}^{*\geqslant 2p^h}_{\mathcal{P}}(\mathrm{Id}^{(h)}, F) = 0$ and $\mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, F)$ is $2p^h$-periodic ; the first period is given by $\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}^{(h)}, F)$*

One can also compare Yoneda module structures.

Remark

$S^n \circ I$ is not in the image of $\mathcal{P}$ by the forgetful functor, but the knowledge of $\mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, S^n \circ I)$ helps calculating $\mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, S^n \circ S^m)$

Remark

Deeper, but incomplete comparisons when first variable is not Id by Franjou, Friedlander, Suslin and Scorichenko.

Definition
A *boolean functor* in $\mathcal{F}$ is an object $B \in \mathcal{F}$ together with an associative and commutative product $* : B \times B \longrightarrow B$ such that $x^{*p} = x$ for each $x \in B(V)$.

Notation
Two products in $S^* \circ B$: xy will denote product in S^* whereas $x * y$ will denote product in B. For example :
- $x_1 * \cdots * x_n$ opposed to $x_1 \cdots x_n$
- x^{*p} opposed to x^p
- x^*_J (product of x_i's for $i \in J$) opposed to x_J
- etc.

Examples of $S(\underline{\lambda})$'s

- $S(1^n) = S^n$
- $S(1^{n-p}p)$ is generated by $x_1 \cdots x_{n-p} y^p$;
  $S(1^{n-p}p) = \mathrm{Ker}(S^n \to SR^n)$;
  Here, SR^n is the *reduced* symmetric power $S^n/(x^p = 0)$.
- If $n = p^h$, then $S((p^{h-k})^{p^k}) = S^{p^k}$;
  Inclusion of S^{p^k} in S^{p^h} is given by power p^{h-k}.
- $n = \sum a_i p^i$, $0 \leqslant a_i < p$; $\underline{\lambda} = \prod (p^i)^{a_i}$;
  Then $S(\underline{\lambda})$ is contained in all others $S(\underline{\mu})$.

Subfunctors of S^n

Definition
p-adic partition of n : partition of the integer n whose parts are all powers of p.

Notation
Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_\ell)$ be a p-adic partition of n. Denote by $S(\underline{\lambda})$ the subfunctor of S^n generated by elements $x_1^{\lambda_1} \cdots x_\ell^{\lambda_\ell}$.

Important fact (Troesch - ?) : The set of p-adic partitions of n is a poset : $\underline{\lambda} \leqslant \underline{\mu}$ iff $\underline{\mu}$ is refined by $\underline{\lambda}$. This poset is a **lattice**.

Proposition
$S(\underline{\lambda}) \subset S(\underline{\mu})$ *iff* $\underline{\lambda} \leqslant \underline{\mu}$,
$S(\underline{\lambda}) \cap S(\underline{\mu}) = S(\min(\underline{\lambda}, \underline{\mu}))$

Motivations – **3. Other related topics**

1. Mac Lane cohomology

Jibladze and Pirashvili proved that if M is a vector space, then $\mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, -\otimes M)$ is equal to $\mathrm{HML}^*(\mathbb{F}_p, M)$. Therefore $\mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, F)$ generalizes Mac Lane cohomology groups.

This situation extands to the case $\mathcal{F}$ is a functor category over R-modules, for any ring R. In this case, M has to be a bimodule.

2. Modular representation of the symmetric groups

$\mathcal{F}_\omega$ has a filtration whose successive quotients are the categories of modular representations of $\mathfrak{S}_n$.

Corollary 1 (former result of Betley)
Poincaré series of $\mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, S^{p^k} \circ I)$: $\displaystyle\prod_{i=1}^{k} \frac{1}{1 - t^{2^i - 1}}$
Yoneda module structure : trivial.

Corollary 2
$\mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, S^{p^k} \circ B) \overset{\text{Yoneda}}{=} \mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, B) \otimes \mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, S^{p^k} \circ I)$
Module structure on the right hand side : from the first factor.

Corollary 3
Series of $\mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, S^{p^{h_m}} \circ \cdots \circ S^{p^{h_1}} \circ I)$: $\displaystyle\prod_{i=1}^{h_1} \frac{1}{1 - t^{2^i - 1}} \cdots \prod_{i=1}^{h_m} \frac{1}{1 - t^{2^i - 1}}$
Yoneda module structure : trivial.

Main results – **2. Decomposition of $S^n \circ B$**

Theorem (Troesch)
Let B be a boolean functor. The Taylor tower of S^n splits after composition by B, that is, $t_{m-1} S^n \circ B$ is a direct summand of $t_m S^n \circ B$. Hence :

$$S^n \circ B = \bigoplus_{\underline{\lambda}} \bigotimes_{i \geqslant 0} SR^{\nu_i(\underline{\lambda})} \circ B,$$

where the sum is taken over all p-adic partitions of n

Similar (easier) result :
The Taylor tower of I also splits after composition by B. Hence :

$$I \circ B = \bigoplus_{n \geqslant 0} SR^n \circ B.$$

Main results – **1. Taylor filtration of S^n**

Theorem
- $t_m(S^n) = \sum_{\underline{\lambda}} S(\underline{\lambda})$,
  (the sum is taken over p-adic partitions of length $\ell \leqslant m$)
- $t_m(S^n)/t_{m-1}(S^n) = \bigoplus_{\underline{\lambda}} \bigotimes_{i \geqslant 0} SR^{\nu_i(\underline{\lambda})}$,
  (the sum is taken over p-adic partitions $\underline{\lambda}$ of length m, and $\nu_i(\underline{\lambda})$ stands for the number of parts of size p^i in $\underline{\lambda}$)
- *If $p = 2$, then $SR^n = \Lambda^n$, hence :*
  $t_m(S^n)/t_{m-1}(S^n) = \bigoplus_{\underline{\lambda}} \bigotimes_{i \geqslant 0} \Lambda^{\nu_i(\underline{\lambda})}$.

Examples of boolean functors

Example
For $p = 2$, every functor factorizing through the category of *boolean algebras* is boolean.

Example
I is boolean : For $\Phi, \Psi : V^* \to \mathbb{F}_p$, $\quad \Phi * \Psi(f) = \Phi(f) \cdot \Psi(f)$.

Example
If B is boolean, so is $\Gamma^n \circ B$. The product is induced by
$$(a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n) \mapsto a_1 * b_1 \otimes \cdots \otimes a_n * b_n.$$

Example
Let S^{p^∞} be the colimit of $S^{p^k} \to S^{p^{k+1}}$. Then S^{p^∞} comes with a commutative, *but not associative* product satisfying $x^{*p} = x$.

Questions and remarks

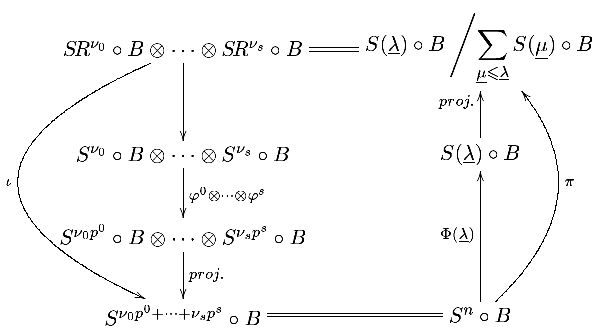
- **For which F's do Taylor towers of $F \circ B$ split ?** All F ?

- I proved that, for $F, G \in \mathcal{P}$ homogeneous of degree p^h and p^k, s.t. $\mathrm{Ext}(\mathrm{Id}, F)$ and $\mathrm{Ext}(\mathrm{Id}, G)$ have trivial module structure :

  $\mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}, G \circ F) = \mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}, F) \otimes \mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}, G) \otimes \mathrm{Ext}^*_{\mathcal{P}}(\mathrm{Id}, S^{p^k} \circ S^{p^h})$.

  Compare with the formula for $\mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, S^n \circ B)$ **!**

  Is there some **more general formula** when $F, G \in \mathcal{F}$ do not lie in the image of $\mathcal{P}$, using $S^n \circ I$, $I \circ S^n$ or $I \circ I$ instead of $S^n \circ S^m$?



A powerful tool : Pirashvili's vanishing theorem

Theorem (Pirashvili)
If F and G have no constant term, $\mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, F \otimes G) = 0$.

Corollary 1
If $n \neq p^h$, $\mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, S^n \circ F) = 0$.

Corollary 2
$\mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, F \circ \Lambda^n \circ G) = \mathrm{Ext}^{*-n+1}_{\mathcal{F}}(\mathrm{Id}, F \circ S^n \circ G)$
$\mathrm{Ext}^*_{\mathcal{F}}(\mathrm{Id}, F \circ \Gamma^n \circ G) = \mathrm{Ext}^{*-n+1}_{\mathcal{F}}(\mathrm{Id}, F \circ \Lambda^n \circ G)$